Problem 1.3.3: A coin is tossed until \( H \) is reached. Let the probability set function \( P \) assign to the space \( \mathcal{C} = \{T^0H, T^1H, T^2H, \ldots \} \) to the values \( P(T^nH) = (\frac{1}{2})^{n+1} \) for each \( n \geq 0 \). Define \( C_1 = \{H, TH, T^2H, T^3H, T^4H\} \) and \( C_2 = \{T^4H, T^5H\} \). Compute \( P(C_2), P(C_1 \cap C_2), \) and \( P(C_1 \cup C_2) \).

**Solution:**

\( C_1 \cap C_2 = \{T^4H\} \) and \( C_1 \cup C_2 = \{H, TH, T^2H, T^3H, T^4H, T^5H\} \).

\[
P(C_1) = \sum_{i=0}^{\infty} P(T^iH) = \sum_{i=0}^{\infty} (\frac{1}{2})^{i+1} = (\frac{1}{2})(\frac{1}{1-\frac{1}{2}}) = \frac{1}{2}.
\]

\[
P(C_2) = \sum_{i=0}^{\infty} P(T^iH) = \sum_{i=0}^{\infty} (\frac{1}{2})^{i+1} = (\frac{1}{2})^5 + (\frac{1}{2})^6 = \frac{3}{64}.
\]

\[
P(C_1 \cap C_2) = P(T^4H) = (\frac{1}{2})^5 = \frac{1}{32}.
\]

\[
P(C_1 \cup C_2) = \sum_{i=0}^{\infty} P(T^iH) = \sum_{i=0}^{\infty} (\frac{1}{2})^{i+1} = (\frac{1}{2})(\frac{1}{1-\frac{1}{2}}) = \frac{63}{64}.
\]

Problem 1.3.4: Let the sample space be \( \mathcal{C} = C_1 \cup C_2 \) with \( P(C_1) = 0.8 \) and \( P(C_2) = 0.5 \). Find \( P(C_1 \cap C_2) \).

**Solution:**

\[
1 = P(\mathcal{C}) = P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2).
\]

Therefore \( P(C_1 \cap C_2) = 0.3 \).

Problem 1.3.9a: Prove the inclusion/exclusion formula for sets \( A, B, C \in \mathcal{C} \).

**Solution:**

\[
P(A \cup B \cup C) = P(A) + P(B \cup C) - P(A \cap (B \cup C))
\]

\[
= P(A) + (P(B) + P(C) - P(B \cap C)) - P((A \cap B) \cup (A \cap C))
\]

\[
= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) + P(A \cap C) - P(A \cap B \cap A \cap C)
\]

\[
= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C).
\]

Problem 1.3.9b: Prove the inclusion/exclusion formula for sets \( \{A_n\} \in \mathcal{C} \).

**Solution:**

This can be proved with induction on the number of sets, \( n \). It is true that for any two sets in \( \mathcal{C} \), the equation holds (per the book). For the induction step, assume that any set of \( n \) subsets \( \{A_1, \ldots, A_n\} \in \mathcal{C} \), the equation holds. Consider now some set \( A_{n+1} \in \mathcal{C} \).

\[
P[\bigcup_{i=1}^{n+1} A_i] = P[\bigcup_{i=1}^{n} A_i \cup (A_{n+1})] = P[\bigcup_{i=1}^{n} A_i] + P[A_{n+1}] - P[\bigcup_{i=1}^{n} (A_i \cap A_{n+1})],
\]

where the first inequality holds by the induction assumption and the second holds by the inclusion/exclusion principle on two sets.

Notice:

\[
P[\bigcup_{i=1}^{n+1} A_i] = \sum_{k_n \leq n} P[A_{k_1}] - \sum_{k_m \leq n} P[A_{k_1} \cap A_{k_2}] + \cdots + (-1)^n \sum_{k_m \leq n} P[A_{k_1} \cap \cdots \cap A_{k_n}]
\]

\[
en + 1 \sum_{A_i \cap A_{n+1}}
\]

Also:

\[
P[\bigcup_{i=1}^{n+1} A_i] = \sum_{k_n \leq n} P[A_{k_1} \cap A_{n+1}] - \sum_{k_m \leq n} P[A_{k_1} \cap A_{k_2} \cap A_{n+1}] + \cdots + (-1)^n \sum_{k_m \leq n} P[A_{k_1} \cap \cdots \cap A_{k_{n-1}} \cap A_{n+1}]
\]

Therefore:

\[
P[\bigcup_{i=1}^{n+1} A_i] = P[A_{n+1}] - P[\bigcup_{i=1}^{n} A_i \cap A_{n+1}]
\]

\[
= \sum_{k_n \leq n} [P[A_{k_1}] + P[A_{n+1}]] - \sum_{k_m \leq n} P[A_{k_1} \cap A_{k_2}] + \sum_{k_m \leq n} P[A_{k_1} \cap A_{n+1}] + \cdots + (-1)^{n+1} P[A_1 \cap \cdots \cap A_{n} \cap A_{n+1}]
\]

\[
= \sum_{k_n \leq n} P[A_{k_1}] - \sum_{k_m \leq n+1} [P[A_{k_1} \cap A_{k_2}] + \cdots + (-1)^{n+2} \sum_{k_m \leq n+1} P[A_{k_1} \cap \cdots \cap A_{k_{n+1}}].
\]

Problem 1.3.12: A person purchased 10 of 1000 tickets sold in a certain raffle. To determine the five prize winners, 5 tickets are to be drawn at random and without replacement. Compute the probability
Furthermore, given 6 pairs of socks, there are 
all combinations of 6 socks taken from 16 socks. 

Therefore, we seek 
\[
P(S^c) = 1 - P(S) = 1 - \frac{|S|}{|O|} = 1 - \frac{7845 \cdot 150 \cdot 698}{8250 \cdot 291 \cdot 250 \cdot 200} = \frac{22 \cdot 507 \cdot 846 \cdot 630}{458 \cdot 349 \cdot 513 \cdot 900} = 0.049106.
\]

Problem 1.3.16: In a lot of 50 light bulbs, there are 2 bad bulbs. An inspector examines 5 bulbs, which are selected at random and without replacement.

a) Find the probability of at least 1 defective bulb among the 5.

b) How many bulbs should be examined so that the probability of finding at least 1 bad bulb exceeds \(\frac{1}{2}\)?

Solution:
\[
\Omega = \{\text{all combinations of 5 bulbs from 50 bulbs}\} \quad |\Omega| =\binom{50}{5} = \frac{50!}{5!(50-5)!} = 2118760.
\]

a) Define \(S = \{\text{all combinations of 5 bulbs that contain no defectives}\}\).

- \(|S| = \binom{48}{5} = \frac{43!}{5!(48-5)!} = 1712304.
- \(P(S) = 1 - \binom{50}{5} = 1 - \frac{48!}{5!(48-5)!} = \frac{29 + 99}{458} = 0.19184.
\]

b) An equivalent statement is "...so that the probability of finding 0 bad bulbs falls below \(\frac{1}{2}\)."

- Define \(S_n = \{\text{combinations of n bulbs from 50 that contain no defectives}\}\).
- \(|S_n| = \binom{48}{n} = \frac{48!}{n!(48-n)!}.
- \(\Omega_n = \{\text{all combinations of n bulbs from 50 bulbs}\} \quad |\Omega_n| = \binom{50}{n} = \frac{50!}{n!(50-n)!}.
- \(\frac{1}{2} \ge P(S_n) = \frac{|S_n|}{|\Omega_n|} = \frac{\binom{48}{n}}{\binom{50}{n}} = \frac{n!(50-n)!}{48!(48-n)!} = \frac{1 \cdot 2 \cdot 3 \cdot 49 \cdot (50-n)}{48!} = \frac{99}{2450} \frac{1}{2} + \frac{99}{2450} = 0.522.
- \frac{1}{2} \ge \frac{99}{2450} n^2 - \frac{99}{2450} n + 1 \implies n \in \left[-\frac{13}{29} \sqrt{29} + \frac{99}{29}, \frac{13}{29} \sqrt{29} + \frac{99}{29}\right] = [14.496, 84.504].
\]

Therefore between 15 and 50 bulbs should be examined.

Problem 1.4.6: A drawer contains 8 different pairs of socks. If six socks are taken at random and without replacement, compute the probability that there is at least one matching pair among these six socks.

Solution:
\[
\Omega = \{\text{all combinations of 6 socks taken from 16 socks}\} \quad |\Omega| = \binom{16}{6} = \frac{16!}{6!(16-6)!} = 8008.
\]

Let \(S = \{\text{all combinations of 6 socks, all from different pairs, taken from 16 socks}\} \quad |S| = \binom{8}{6} = \frac{8!}{6!(8-6)!} = 28.

Notice we want to calculate \(P(S^c)\).

For each element in \(S, 6\) of the 8 pairs must be in the set. There are \(\binom{8}{6}\) ways to pick 6 of 8 pairs. Furthermore, given 6 pairs of socks, there are \(2^6\) unique selections.

So \(|S| = \binom{8}{6} \cdot 2^6 = 8008 \cdot 2^6 = 1792.\)

Therefore \(P(S^c) = 1 - P(S) = 1 - \frac{|S|}{|\Omega|} = 1 - \frac{1792}{8008} = \frac{111}{143} = 0.77622.
\]

Problem 1.4.8: Machines I, II, and III are all producing springs of the same length – producing 1\%, 4\%, and 2\% defective springs, respectively. Of the total production of springs in the factory, they produce 30\%, 25\%, and 45\%, respectively.

a) If a spring is selected at random from the total springs produced in a given day, determine the probability that it is defective.

b) Given that the selected spring is defective, find the conditional probability that it was produced by Machine II.

Solution:
Let \(D, T, H\) be the events for a spring being defective and non-defective, respectively.

Let \(O, T, H\) be the events for a spring coming from machine 1, 2, and 3 respectively.

\[
P[D|O], P[D|T], P[D|H], P[O], P[T], \text{ and } P[H] \text{ are given in the problem.}
\]

a) This question seeks \(P[D] = P[D \cap O] + P[D \cap T] + P[D \cap H] = P[D|O] \cdot P(O) + P[D|T] \cdot P(T) + P[D|H] \cdot P(H) = (0.01)(0.3) + (0.04)(0.25) + (0.02)(0.45) = 0.022.
\]

b) This question seeks \(P[T|D] = \frac{P[T \cap D]}{P[D]} = \frac{P[D|T] \cdot P(T)}{P[D]} = \frac{0.04(0.25)}{0.022} = 0.45455.
\]
Problem 1.4.10: Two boxes of computer disks. Box $C_1$ contains 7 verbatim disks and 3 control data disks. Box $C_2$ contains 2 verbatim disks and 8 control data disks. She chooses a box at random with $P[C_1] = \frac{2}{3}$ and $P[C_2] = \frac{1}{3}$. A disk is then selected at random and the event $C$ occurs if it is from Control Data. Using an equally likely assumption for each disk in the selected box, compute $P[C_1|C]$ and $P[C_2|C]$.

Solution:
Given in the problem are $P[C|C_1] = \frac{1}{2}$, $P[C|C_2] = \frac{8}{10}$, $P[C_1] = \frac{2}{3}$, and $P[C_2] = \frac{1}{3}$. 

\[
P[C_1|C] = \frac{P[C|C_1] P[C_1]}{P[C]} = \frac{\frac{1}{2} \cdot \frac{2}{3}}{\frac{2}{3} \cdot \frac{1}{2}} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 3 \cdot \frac{1}{6} = 0.42857.
\]

\[
P[C_2|C] = \frac{P[C|C_2] P[C_2]}{P[C]} = \frac{\frac{8}{10} \cdot \frac{1}{3}}{\frac{2}{3} \cdot \frac{1}{2}} = \frac{0.8}{0.6} = 4 \cdot \frac{1}{6} = 0.57143.
\]

Problem 1.4.12: Let $C_1$ and $C_2$ be independent events, with $P[C_1] = .6$, $P[C_2] = .3$. Compute $P[C_1 \cap C_2]$, $P[C_1 \cup C_2]$, and $P[C_1 \cup C_2^c]$.

Solution:

\[
P[C_1 \cap C_2] = P[C_1] \cdot P[C_2] = (0.6)(0.3) = 0.18.
\]

\[
P[C_1 \cup C_2] = P[C_1] + P[C_2] - P[C_1] \cdot P[C_2] = (0.6) + (0.3) - (0.6)(0.3) = 0.72.
\]

\[
P[C_1 \cup C_2^c] = P[C_1] + P[C_2] - P[C_1] \cdot P[C_2] = (0.6) + (0.7) - (0.6)(0.7) = 0.88,
\]

where the first inequality in the last line occurs because $C_1$, $C_2$ independent implies $C_1$, $C_2^c$ independent.

Problem 1.5.2: Let $X$ be a random variable with pdf $p(x)$. Find the constant $c$.

a) $p(x) = c \left( \frac{2}{3} \right)^x$ for $x \in \mathbb{N}_+$.

b) $p(x) = cx$ for $x = 1, 2, 3, 4, 6$, and 0 elsewhere.

Solution:

a) $1 = \sum_{x=1}^{\infty} \left( \frac{2}{3} \right)^x = c \left( \frac{2}{3} \right) \sum_{x=0}^{\infty} \left( \frac{2}{3} \right)^x = c \left( \frac{1}{1 - \frac{2}{3}} \right) = 2c \implies c = \frac{1}{2}$.

b) $1 = \sum_{x=1}^{6} cx = c(1 + \cdots + 6) = c(21) \implies c = \frac{1}{21}$.

Problem 1.5.6: Let $P_X(D) = \int_{D} \! f(x) \, dx$, for $f(x) = \frac{2x}{9}$ and $x \in (0, 3)$.

Let $D_1 = (0, 1)$ and $D_2 = (2, 3)$. Compute $P_X(D_1)$, $P_X(D_2)$, and $P_X(D_1 \cup D_2)$.

Solution:

\[
P_X(D_1) = \int_{(0,1)} \frac{2x}{9} \, dx = \left[ \frac{x^2}{9} \right]_0^1 = \frac{1}{9}.
\]

\[
P_X(D_2) = \int_{(2,3)} \frac{2x}{9} \, dx = \left[ \frac{x^2}{9} \right]_2^3 = \frac{2}{9}.
\]

\[
P_X(D_1) = \int_{(0,1) \cup (2,3)} \frac{2x}{9} \, dx = \int_{(0,1)} \frac{2x}{9} \, dx + \int_{(2,3)} \frac{2x}{9} \, dx = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}.
\]

Problem 1.5.8: Define $F(x) = \frac{x+2}{4}$ for $x \in [-1, 1]$. 0 and 1 trivially elsewhere.

Sketch $F(x)$ and compute:

a) $P[-\frac{1}{2} < X \leq \frac{1}{2}]$

b) $P[X = 0]$

c) $P[X = 1]$

d) $P[2 < x \leq 3]$.

Solution:

. . .

. . .

a) $P[-\frac{1}{2} < X \leq \frac{1}{2}] = F(\frac{1}{2}) - F(-\frac{1}{2}) = \frac{\frac{1}{2} + 2}{4} - \frac{-\frac{1}{2} + 2}{4} = \frac{1}{4} = 0.25$.

b) $P[X = 0] = 0.$

c) $P[X = 1] = \frac{1}{2} = 0.25$.

d) $P[2 < x \leq 3] = 0.$

Problem 1.6.2: Bowl contains 10 chips, with one red. Draw until a red is chosen without replacement. Define $X$ = number of draws needed to pull red chip.

Find the pmf of $X$ and $P[X \leq 4]$.

Solution:

$P[X = 1] = \frac{1}{10}$.
\[ P[X = 2] = \frac{9}{10} \cdot \frac{1}{6} = \frac{9}{60} \]
\[ P[X = k] = \left( \frac{9}{10} \cdot \frac{8}{6} \cdots \frac{11-k}{11-k} \right) \left( \frac{1}{11-k} \right) = \frac{1}{10} \text{ for } k \geq 3. \]

Therefore \( P[X = k] = \frac{1}{10} \) for \( k = 1, \ldots, 10. \)

Resulting in \( P[X \leq 4] = \sum_{i=1}^{4} P[X = i] = \frac{4}{10}. \)

**Problem 1.6.3:** Cast a die a number of independent times until a six appears on the up side.

a) Find the pmf \( p(x) \) of \( X \), the number of casts needed to obtain the first six.
b) Show that \( \sum_{x=1}^{\infty} p(x) = 1. \)
c) Determine \( P(X = \text{odd}). \)
d) Find the cdf \( F(x) \).

**Solution:**

a) \( p(x) = \left( \frac{5}{6} \right)^{x-1} \left( \frac{1}{6} \right) = \frac{1}{6} \cdot \frac{5}{6}^x. \)
b) \( \sum_{x=1}^{\infty} p(x) = \left( \frac{1}{6} \right) \sum_{x=0}^{\infty} \left( \frac{5}{6} \right)^x = \frac{1}{6} \cdot \frac{1}{1-\frac{5}{6}} = 1. \)
c) \( P(X = \text{odd}) = \frac{1}{6} \cdot \left( \frac{5}{6} \right)^0 + \frac{1}{6} \cdot \left( \frac{5}{6} \right)^2 + \cdots = \frac{1}{6} \sum_{k=0}^{\infty} \left( \frac{5}{6} \right)^{2k} = \frac{1}{6} \cdot \frac{1}{1-\left( \frac{5}{6} \right)^2} = \frac{6}{11}. \)
d) \( F(k) = P(X \leq k) = \sum_{x=1}^{k} p(x) = \left( \frac{1}{6} \right) \sum_{x=0}^{k-1} \left( \frac{5}{6} \right)^x = \frac{1}{6} \cdot \frac{1-\left( \frac{5}{6} \right)^k}{1-\frac{5}{6}} = 1 - \left( \frac{5}{6} \right)^k. \)