Monte Carlo Valuation
Black-Scholes formula

\[ S_0 \]

\[ S_T - K \]

\[ K \]

\[ 0 \]

\[ S_T = \ln \left( \frac{S_T - K}{K} \right) \]

\[ C_T = \max \left( S_T - K, 0 \right) \]

\[ C_0 = e^{-rT} E(C_T) \]

Monte Carlo valuation = do the \( E(\cdot) \) by simulation.
Monte Carlo Valuation

Simulate $S_t \sim LN$

for each realization of $S_t$, calculate $C_t$.

Average $C_t$ after many iterations.

$$\overline{C_t} \approx E(C_T)$$

$$C_0 = e^{-rT} \overline{E(C_T)}$$
Risk-neutral view

\[ S_T \sim LN \left( (r - \delta - \frac{1}{2} \sigma^2)T, \sigma^2 T \right) \]

\[ \Rightarrow E(S_T) = e^{(r - \delta)T} \]

- Monte Carlo valuation using A holds risk-neutral view of the market.

- It is possible to compute MC with true probability, but much more demanding computationally.
Pros of MC valuation

- B-S only computes \( C_T \) at time \( T \).
- Some options are path dependent. (Asian option)
- Use MC to simulate each path.
Simulating Lognormal Random Variable

\[ S_t \sim \ln N(\mu, \sigma^2) \]

\[ S_t = e^x \quad x \sim N(\mu, \sigma^2) \]

\[ x = \mu + \sigma^2 \cdot z \quad z \sim N(0,1) \]

\[ S_t = e^{\ln S_0 + (r - \delta - \frac{1}{2} \sigma^2)T + \sigma \sqrt{T} \cdot z} \]

\[ = S_0 \cdot e^{(r - \delta - \frac{1}{2} \sigma^2)T + \sigma \sqrt{T} \cdot z} \]
Monte Carlo Valuation

\[ C_0 = e^{-rT} \left[ \frac{1}{n} \sum_{i=1}^{n} C_i^* \right] \]

\[ C_T^i = \max \left( 0, \ S_T^i - K \right) \]

\[ S_T^i = S_0 \ e^{\left( \left( r - \frac{\sigma^2}{2} \right) T + \sigma \sqrt{T} \cdot Z^i \right) + \sigma \cdot \text{simulated} \ N(0,1)} \]
Arithmetic Asian Option with MC

\[ T = 3 \text{ mo.} \]

Average value of 1st, 2nd, and 3rd mo.

\[ C_{\text{Asian}} = e^{-rT} \mathbb{E} \left[ \max \left( \frac{S_1 + S_2 + S_3}{3} - K, 0 \right) \right] \]
Accuracy of Monte Carlo

How close \( \frac{1}{n} \sum_{i=1}^{n} c_i \) is to \( E(C_T) \)?
By CLT
\[
\frac{1}{n} \sum_{i=1}^{n} c_i^t \sim N \left( \mu(c_t), \frac{v(c_t)}{n} \right)
\]

\[\bar{c}_t\]

use sample SD of \(c_t^i\).

\[\bar{c}_t \text{ is within } E(c_t) \pm 1.96 \frac{SD(c_t)}{\sqrt{n}} \text{ 95\% of the time.}\]

If \(SD(c_t) = 4.05 \) and \(c_0 = \$3\), and you want 1\% accuracy \((0.03)\), then you need

\[1.96 \frac{4.03}{\sqrt{n}} = 0.03 \quad \Rightarrow \quad n \approx 70000 \text{ iterations}\]
Efficient Monte Carlo Valuation

$\bar{C}_T \rightarrow E(C_T)$ as

$\bar{C}_T \sim N( E(C_T), \frac{\sigma(C_T)}{n} )$

Can I do better?

(faster)
Antithetic Variates

Calculate $\tilde{C}_t$, $\tilde{C}_t$ where $\tilde{C}_t$ and $\tilde{C}_t$ are negatively correlated.

Then let

$$C_T = \frac{\tilde{C}_t + \tilde{C}_t}{2}$$

and

$$E(C_T) = C_T$$
\[
\text{Var} \left( \frac{\hat{C}_T^i + \tilde{C}_T^i}{2} \right) = \frac{1}{4} \left[ \text{Var} \left( \hat{C}_T^i \right) + \text{Var} \left( \tilde{C}_T^i \right) + 2 \text{Cov} \left( \hat{C}_T^i, \tilde{C}_T^i \right) \right]
\]

\[
\text{Var} \left( \hat{C}_T^i \right) \quad \text{and} \quad \text{Var} \left( \tilde{C}_T^i \right) \quad \text{neg.}
\]

\[
\frac{\text{Var} \left( C_T^i \right)}{2} < \frac{\text{Var} \left( C_T^i \right)}{2} \quad \text{Arithmetic with } \frac{n}{2} \text{ each}
\]

\[
\frac{\text{Var} \left( C_T^i \right)}{2} \quad \text{Regular MC with } \frac{n}{2}
\]
Control Variates

\[ C_T, \quad \text{const} \]

\[ C_T^* = C_T + c \left( T - \mu_T \right) \]

\[ \mathbb{E}(T) = \mu_T \]

\[ \text{Var}(C_T^*) = \text{Var}(C_T) + c^2 \text{Var}(T - \mu_T) + 2c \text{Cov}(C_T, T - \mu_T) \]

\[ = \text{Var}(C_T) + c^2 \text{Var}(T) + 2c \text{Cov}(C_T, T) \]

What \( c^* \) minimize \( \text{Var}(C_T^*) \)?
Take $\frac{d}{dc}$ and set to 0.

$$2C \text{Var}(T) + 2 \text{Cov}(C_T, T) = 0$$

$$C = \frac{\text{Var}(C_T)}{\text{Var}(T)} - \frac{\text{Cov}(C_T, T)}{\text{Var}(T)}$$

Put it back in, we get,

$$\text{Var}(C_T^+) = \text{Var}(C_T) + \left(\frac{\text{Cov}(C_T, T)}{\text{Var}(T)}\right)^2 \cdot \text{Var}(T) + 2 \frac{-\text{Cov}(C_T, T)}{\text{Var}(T)} \cdot \text{Cov}(C_T, T)$$
\[ \text{Var}(c_T^k) = \text{Var}(c_T) - \frac{\text{Cov}(c_T, T)}{\text{Var}(T)} \]

\[ \frac{\text{Cov}(c_T, T)}{\sqrt{\text{Var}(c_T) \cdot \text{Var}(T)}} = \text{Corr}(c_T, T) = 8 \]

\[ \text{Var}(c_T^k) = \text{Var}(c_T) \left[ 1 - 8^2 \right] \]

Pick \( T \) so that \( 8^2 \) is close to 1.
\( C_t^* = C_t^i + -\left( \frac{Cov(C_t, T)}{Var(T)} \right) \left[ T^i - \mu_T \right] \)

\[ \uparrow \quad \text{Compute or estimate} \]

\( E(C_t) \approx \overline{C_t^*} \)
Lognormal Stock Model with Poisson Jumps

\[ S_T \sim LN(\ln(S_0) + (\nu - \frac{1}{2} \sigma^2)T, \sigma^2 T) \]

Tail distribution is too little

Often you see some large values here.
Modify Lognormal Model as...

Original model:

\[ S_t = S_0 e^{(r - \frac{1}{2} \sigma^2) h + \sigma \sqrt{h} \cdot Z} \]

\( Z \sim N(0,1) \)

Modified model:

\[ S_t = \left[ S_0 e^{(r - \frac{1}{2} \sigma^2) h + \sigma \sqrt{m} \cdot Z} \right] e^{m \left( a_j - \frac{1}{2} \sigma_j^2 \right) + \sigma_j \sum_{i=1}^{m} W_i(i)} \]

\( m \sim Poisson(\lambda) \)

\( W_i(i) \sim N(0,1) \)

Jump component

\( S_t \) is still lognormally distributed.
Since the jump component is lognormally distributed, new $S_t$ is still lognormal, (proof of LN)

$M = \# \text{ of jumps}$,

$m \sim \text{Poi}(\lambda k)$, where $k = \text{av. } \# \text{ of jumps per year}$.

Because of Poisson, $M$ is often 0 (no jumps).
Simulated stock price paths over 10 years (3650 days). One stock cannot jump; the other is the same except that jumps can occur. The simulation assumes that $\alpha = 8\%$, $\delta = 0$, $\sigma = 30\%$, $\lambda = 3$, $\alpha_f = -2\%$, and $\sigma_f = 5\%$. 
Figure 19.8

Histograms and normal probability plots for the daily returns generated from the two series in Figure 19.7. Graphs on the left are for the no-jump series.