Forwards and Futures
Types of Forwards

Outright Purchase
\[ \text{Pay at } t = 0 \quad \text{Receive Asset at } t = 0 \quad \text{Price} \quad S_0 \]

Fully leveraged Purchase
\[ \text{Pay at } t = T \quad \text{Receive Asset at } t = 0 \quad \text{Price} \quad S_0 e^{rT} \]

Prepaid forward
\[ \text{Pay at } t = 0 \quad \text{Receive Asset at } t = T \quad \text{Price} \quad ? \]

Forward
\[ \text{Pay at } t = T \quad \text{Receive Asset at } t = T \quad \text{Price} \quad (?).e^{rT} \]
Prepaid Forwards

Pay at \( t=0 \), receive at \( t=T \).

- Could miss dividends.

If there are no dividends to be paid,

\[
F_{0,T} = S_0.
\]  (paid at \( t=0 \))
\[ r = \text{Force of interest} \]

\[ A(t) = A(0) e^{\int_r^t dt} = A(0) e^{rt} \]

**Present Value**

\[ A(0) = A(t) e^{-\int_r^0 dt} = A(t) e^{-rt} \]
IF $F_{0,T}^P \neq S_0$, then there's an arbitrage

Suppose $F_{0,T}^P \neq S_0$, say $A_0$. Then you can

$A_0 \geq S_0$

$t = 0$

Sell forward at $A_0$. \[ t = T \]

Buy same stock at $-S_0$. \[ \rightarrow S_T \]

$A_0 < S_0$

Buy prepaid forward $-A_0$. \[ \rightarrow S_T \]

Short sale stock $S_0$. \[ -S_T \]
Prepaid Forward with Dividends

Discrete dividends

\[ F_{0,T} = S_0 - \sum_{i=1}^{n} PV(D_{t_i}) \]

\[ D_{t_i} = \text{dividend paid at} \quad t = t_i \]
Ex 5.1

\[ S_0 = 100. \]

\$1.25 dividend quarterly. 1st one in 3 mo.

annual risk free rate 10%. Continuously compounded.

1-year paid forward price?
Review.

Nominal annual rate \( i^{(n)} = 8\% \).

Effective annual rate \( 1 + i = \left(1 + \frac{i^{(n)}}{m}\right)^m \).

\( m = 2 \quad 1 + i = 1.0816 \)

\( m = 3 \quad 1 + i = 1.0824 \)

\( m = \infty \quad 1 + i = \lim_{m \to \infty} \left(1 + \frac{i^{(n)}}{m}\right)^m = e^{i^{(n)}} = e^{0.08} = 1.0833 \).

Force of Interest \( \gamma = \ln(1 + i) = \ln(e^{i^{(n)}}) = i^{(\infty)} \).

\( n = \infty \).
Ex 5.1

Continuously compounded
(nominal) annual rate

\[ i^{(1)} = 10\% \]

Continuously compounded
nominal quarterly rate

\[ j^{(4)} = ? \]

Annual

\[ \int_0^1 e^{-i t} \, dt = e^{-i} = 1 + i \]

Quarterly

\[ \int_0^1 e^{j t} \, dt = e^{j} = 1 + j = (1 + i)^{1/4} \]

effective quarterly
Continuously compounded

Annual rate

Quarterly rate

Annual 10%  

\[ e^r = (1+i)^4 \]

\[ r = \frac{10\%}{4} = 2.5\% \]
Actual rate

\[
\hat{F}_{0, \tau}^- = 1000 - \sum_{i=1}^{4} 1.25 \cdot e^{-0.25(i/4)}
\]

Quarterly rate

\[
\hat{F}_{0, \tau}^- = (1000 - \sum_{i=1}^{4} 1.25 \cdot e^{-0.25(i)})
\]

\[
\hat{F}_{0, \tau}^- = 95.30
\]
Price of Prepaid Fund with continuous dividends

+ approximation

Dividend yield is fixed.

Daily dividend \( \frac{\delta}{365} \) So

\( \delta : \) annual dividend yield.

\( \frac{1}{i(365)} \)

If we reinvest all the dividends,

\[ A(t) = A_0 \left(1 + \frac{\delta}{365}\right)^{365t} \rightarrow A_0 e^{\delta t} \]

\( t = \gamma(m) \)
\[ F_{0,T}^e = S_0 \, e^{-rT} \]

\text{w/ cont. dividends}
Forward Contract

Pay at $t=T$, receive at $t=T$.

When there's no dividends:

$$F_{0,T} = FV(S_0) = S_0 e^{rt}$$

$r =$ risk-free rate
Continuously compounded.
If forward price is not $S_0 e^{rt}$, there's arbitrage.

\[ \text{Say } F_{0,T} = A_0 \neq S_0 e^{rt} \]

\[ A_0 > S_0 e^{rt} \quad \text{at } t=0 \quad \text{and } t=T \]

- sell forward at $A_0$
- borrow at rate $r$
- buy asset

\[ S_0 \rightarrow -S_0 e^{rt} \]

\[ 0 \quad A_0 - S_0 e^{rt} \]
\[ A_0 < S_0 e^{rT} \]

\[ t=0 \quad t=T . \]

- **Buy forward**: 0 \[ \rightarrow \] \(-A_0\)

- **Lend at rate** \( r \): \(-S_0 \rightarrow S_0 e^{rT}\)

- **Short asset**: \( S_0 \quad \text{asset will come from forward.} \)
\[ F_{0,T} = S_0 e^{rT} - \sum_{i=1}^{n} e^{(r-t_i)} P_{ti} \]

**Continuous Dividends**

\[ F_{0,T} = S_0 e^{(r-g)T} = \frac{e^{rT}}{e^{-gT}} \]

PV or stock, no dividends.
Futures Contracts
Futures

almost like forwards.

Agreement to buy/sell asset at given price in future dates.
Differences from forwards

- Standardized - dates, locations, procedures, quantity.

- Settled daily

- There's daily price limits in future markets.

  e.g., if S&P 500 > more than 5%, then trading stops for some period.
Example

SP500 futures at Chicago Mercantile Exchange

Size $250 \times SP500 \text{ index} = \text{dollars}

Cash settled based on SP500 price on 3rd Friday.

Suppose you want to long SP500 futures for $2.2 million.

SP500 today = 1100

1 contract = $250 \times 1100 = $275 \text{ mil.}

$2.2 \text{ mil} = 8 \text{ contracts.}

Broker finds somebody to sell this to you. (short)
Broker will require 'margin' from both buyer and seller.

It next day price = 1099,

\[8 \text{ contracts} = 8 \times 250 \times 1100 = 2,200,000\]

must pay for $2500

but $2500 only worth

\[8 \times 250 \times 1099 = 2,198,000\]

Margin will acquire interest.