Section 7.2: Hamilton circuits and traveling salesmen: Efficient routes

A Hamilton circuit in a graph is a circuit that visits each vertex exactly once.

Example 1: Identify two Hamilton circuits in this graph:

Vertices of Degree 2 and Hamilton Circuits
If a graph has a vertex of degree 2, then each edge meeting that vertex must be part of any Hamilton circuit.

Example 2: Delivery route. Find a Hamilton circuit in the graph to the right.
**Example 3:** Figure 7.69 shows hiking trails and points of interest in Yellowstone National Park. We want to find a path that starts and ends at Shoshone Lake and visits each point of interest exactly once—i.e., we want to find a Hamilton circuit. Either find a Hamilton circuit or explain why no such circuit exists.

**Hamilton Circuits**

1. A Hamilton circuit is a circuit that visits each vertex exactly once.
2. There is no set procedure for determining whether Hamilton circuits exist. *Here is one helpful observation: If a graph has a vertex of degree 2, then each edge meeting that vertex must be part of any Hamilton circuit.*
3. A Hamilton circuit cannot contain a smaller circuit.

In a **complete graph**, each vertex is connected to every other vertex by an edge. The **traveling salesman problem** applies to complete graphs for which a distance is assigned to each edge. The problem is to find a **shortest** Hamilton circuit.
Example 4: The graph in Figure 7.73 shows a delivery map for a trucking firm based in Kansas City.

The firm needs a shortest route that will start and end in Kansas City and make stops in Houston, Phoenix, and Portland. That is, the trucking firm needs a solution of the traveling salesman problem for this map. Calculate the mileage for each possible route to find the solution.

![Delivery Map](image)

FIGURE 7.73 Delivery map.

A complete graph on $n$ vertices has $\frac{(n-1)!}{2}$ distinct Hamilton circuits; this number grows very quickly as $n$ increases, making it impractical to find the absolute best solution to the traveling salesman problem in a large graph. Thankfully, there are efficient algorithms that produce approximate solutions.

The nearest-neighbor algorithm constructs a Hamilton circuit in a complete graph by starting at a vertex. At each step, it travels to the nearest vertex not already visited (except at the final step, where it returns to the starting point). If there are two or more vertices equally nearby, any one of them may be selected.

Example 5: We run a trucking company that makes deliveries in Seattle, Minneapolis, Buffalo, Memphis, and San Diego. Use the nearest-neighbor algorithm to find an approximate solution of the traveling salesman problem.
Example 6: The local school board needs to visit the high school, middle school, primary school, and kindergarten each day. Use the nearest-neighbor algorithm starting at the office to approximate a shortest Hamilton circuit.

Another method used to find approximate solutions of the traveling salesman problem is the cheapest link algorithm: begin by selecting the shortest edge in the graph, and continue by selecting the shortest edge that has not already been chosen, following these rules:
1. Do not choose an edge that results in a circuit except at the final step, when a Hamilton circuit is constructed.
2. Do not choose an edge that would result in a vertex of degree 3.

Example 7: Use the cheapest link algorithm to approximate a shortest Hamilton circuit for the school board in the previous example.

Example 8: Use the cheapest link algorithm to find an approximate solution of the traveling salesman problem for the trucking company in Example 5.

The Traveling Salesman Problem
1. A complete graph is one in which each pair of vertices is joined by an edge.
2. If a distance (more generally, a value) is assigned to each edge, the traveling salesman problem is to find a shortest Hamilton circuit.
3. Solving the traveling salesman problem is difficult when there are more than just a few vertices. However, there are algorithms that can give an approximate solution. One such is the nearest-neighbor algorithm. Another is the cheapest link algorithm.

Practice Problems: 1, 3, 5, 9, 11, 15, 21, 23, 25, 31, 37