Uncertainty Analysis

- Uncertainty is the **probable error range**
- True value for a measurement is unknown but we can use statistics to determine an error range with a certain range of confidence (e.g. 95%)
- Uncertainty Analysis are error bands, which are most likely not exceeded
- Uncertainty does not include simple mistakes such as wrong conversion factors, misreading instruments, …
Measurement Errors

- Distinguish between precision and accuracy:
  - **Precision** (=random error) can be established through repeated measurements
  - **Bias** (=systematic error) requires a reference measurement
- **Accuracy** is good precision and small bias

Dart Board Example: Precision, Bias and Accuracy
Measurement Errors

- Total error is sum of precision and bias error
- Bias error shifts the sample mean away from the true mean

- Precision error creates a normal distribution of the measured data around the sample mean
Uncertainty

• To estimate the true value we define an uncertainty $u_x$ range around the sample mean $\bar{x}$

$$x' = \bar{x} \pm u_x$$

True Mean $\pm$ Sample Mean $\pm$ Uncertainty Interval

• The uncertainty is an error band, which contains the true mean within a certain probability (95%)

• Uncertainty analysis is the method to quantify the term $u_x$
Combining Errors
Root Sum Squares (RSS) Method

• Multiple error sources such as conversion errors, linearity errors, noise, drift combine to form the total measurement error.

• Strategy to combine errors is based on the RSS Method.

\[ u_x = \pm \sqrt{u_1^2 + u_2^2 + \ldots} \quad (P = 95\% \text{ probability}) \]

• In the above equation it is important that all error terms are based on the same units and same reference.
Example

A 26 lb calibrated weight (accurate to within 0.1%) is placed on a scale. 20 repeated measurements are performed and the data is shown on the right.

A statistical analysis of the data gives a mean value $\bar{x}$ of 25.72 and a sample standard deviation $S_x = 0.34$.

Estimate the uncertainty of the mass measurement $u_m$.

**Cal Error**

$$m_{cal} = m_{cal-nominal} \pm u_{cal} = 26 \pm 0.026 \text{ lb}_m \text{ (Prob. = 95%)}$$

$$u_{cal} = 0.026 \text{ lb}_m$$

**Prec. Error**

$$u_{Precision} = t_{v,P} S_x = t_{19,95\%} S_x = 2.093 \times 0.34 = 0.71 \text{ lb}_m$$

**Bias Error**

$$u_{bias} = 26 - 25.72 = 0.28 \text{ lb}_m$$

**Total Error**

$$u_m = \sqrt{u_{cal}^2 + u_{bias}^2 + u_{Precision}^2} = \sqrt{0.026^2 + 0.28^2 + 0.71^2} = 0.77 \text{ lb}_m$$

**Bias Removed**

$$u_m = \sqrt{u_{cal}^2 + 0 + u_{Precision}^2} = \sqrt{0.026^2 + 0 + 0.71^2} = 0.71 \text{ lb}_m$$
Design Stage Uncertainty

- Design Stage Uncertainty deals with the uncertainty during the design of an experiment
- Can be used to determine the uncertainty to be expected in a measurement
- Useful in selecting instrumentation and measurement techniques
Zero Order Uncertainty (Instrument Resolution Error)

- All instruments have a finite resolution
- Zero Order Uncertainty $u_0$ associated with an instrument is defined to be $\frac{1}{2}$ of the resolution

$$u_0 = \pm \frac{1}{2} \text{resolution} \quad (P = 95\% \text{ probability})$$

Example: A 12 bit data acquisition board is configured for a Voltage range of 0-10V. What is the uncertainty $u_0$ associated with the resolution?

$$\text{Res} = \frac{10V}{2^{12}} = 2.44mV$$

$$u_0 = \pm \frac{\text{Res}}{2} = 1.22mV$$
Other Instrument Errors

- Other errors may be linearity, drift, temperature
- Usually a calibration sheet is provided with each instrument, which specifies the instrument precision

Example: The specification for a 12-bit data acquisition board lists the following errors:

- **linearity error**: 0.01% of full range (0-10V)
- **repeatability error**: 3 LSB (Least Significant Bits, digitized number can be off by up to 3 bits)

What is the uncertainty $\mu_c$ associated with this instrument?
The **linearity error** $\mu_1$ is given as a percentage at full range. Convert to Volts:

$$\mu_1 = 0.01\% \times 10V = 1mV$$

The **repeatability error** is given as a bit (LSB) error. Convert to Volts using the board resolution:

$$\mu_2 = \frac{3\text{bits}}{4096\text{bits}} \times 10V = 7.3mV$$

The total instrument error $\mu_c$ is found by combining the two errors with the RSS method:

$$\mu_c = \sqrt{\mu_1^2 + \mu_2^2} = \sqrt{7.3^2 + 1^2} \text{ mV} = 7.4mV$$

For this instrument, the repeatability error clearly dominates the instrument error.
Total Design Stage Uncertainty

- Combination of zero order uncertainty and instrument error
- Total design stage uncertainty is the lowest uncertainty, which can be achieved under the best circumstances
- Real test conditions usually increase the uncertainty due to environmental influences (temperature, temporal and spatial fluctuations)

\[ u_d = \sqrt{u_0^2 + u_c^2} \]

**Example:** Calculate the total design state uncertainty for the 12-bit data acquisition board using the zero-order uncertainty and the instrument error from the earlier examples:

\[ u_d = \sqrt{u_0^2 + u_c^2} = \sqrt{7.4^2 + 1.22^2} \text{ mV} = 7.5 \text{ mV} \]

The total design stage uncertainty for this data acquisition board is 7.5 mV. It is primarily due to the 3-bit repeatability error of the board.
Sources of Errors

- **Calibration Errors:**
  - Bias and precision errors in the calibration standard (the calibration is only as good as the standard used to calibrate the instrument)
  - Application of the standard to the measurement system (how is the calibration performed)

- **Data Acquisition Errors:**
  - Instrument errors (signal conditioning, digitization, see also earlier examples)
  - Uncontrolled variables such as environmental changes, power fluctuations, spatial variation, …

- **Data Reduction Errors:**
  - Truncation error of calculations
  - Curve fit errors

When trying to identify errors, it is useful to first try to identify possible error sources based on the above groupings.
Bias (Systematic) and Precision (Random) Error

- An error may contain either bias (systematic) or precision (random) errors or both.
- An error is considered a precision error, if it is obtained from a statistical estimation only. Otherwise the error is considered a bias error.
- **Bias** errors cannot be found through repeated measurements. They are systematic.
- To estimate a bias error, we need a reference for comparison such as a calibration process, comparison of tests between laboratories, multiple methodologies or experience.
Propagation of Error

- How does an error propagate from input \( x \) to output \( y \)?

\[
\bar{y} = f(\bar{x})
\]

\[
\bar{y} \pm u_y = f(\bar{x} \pm u_x)
\]

\[
\bar{y} \pm u_y \approx f(\bar{x}) \pm \frac{\partial f}{\partial x}|_{x=\bar{x}} \cdot u_x
\]

\[
u_y = \frac{\partial f}{\partial x}|_{x=\bar{x}} \cdot u_x
\]
Propagation of Error
(mult. Variables)

- How does an error propagate for several variables $x_i$ to output $y$?
- Use RSS Method to combine errors

$$u_y = \left[ \left( \frac{\partial y}{\partial x_1} \bigg|_{x_1=x_1} \right)^2 u_{x_1}^2 + \left( \frac{\partial y}{\partial x_2} \bigg|_{x_2=x_2} \right)^2 u_{x_2}^2 + ... \right]^{1/2}$$

$$\frac{u_y}{y} = \left[ \left( \frac{\partial y}{\partial x_1} \bigg|_{x_1=x_1} \right)^2 \frac{u_{x_1}^2}{y^2} + \left( \frac{\partial y}{\partial x_2} \bigg|_{x_2=x_2} \right)^2 \frac{u_{x_2}^2}{y^2} + ... \right]^{1/2}$$

$$e_y = \sqrt{e_1^2 + e_2^2 + ... + e_i^2}$$

$$e_i = \frac{1}{y} \frac{\partial y}{\partial x_i} \bigg|_{x_i=x_i} u_{x_i}$$
Quick Calculation of Rel. Uncertainty for special function \( f(x_i) \)

Error Propagation Function

\[
y = f(x_1, \ldots, x_i, \ldots) = g(x_1, \ldots)x_i^n
\]

\[
\frac{\partial y}{\partial x_i} = \frac{\partial (g(x_i^n))}{\partial x_i} = g(x_i^n)x_i^{n-1}
\]

\[
e_i = \left. \frac{1}{y} \frac{\partial y}{\partial x_i} \right|_{x_i = x_i} u_x_i
\]

\[
e_i = \left. \frac{1}{y} \frac{\partial y}{\partial x_i} \right|_{x_i = x_i} u_x_i = \frac{1}{g(x_i^n)} g(x_i^n)x_i^{n-1} u_x_i = \frac{nx_i^{n-1}}{x_i^n} u_x_i = n \frac{u_{x_i}}{x_i}
\]

Relative Uncertainty
Strategy for Uncertainty Calculation

1. Draw a **diagram** of the test system and label the signals
2. Determine the **nominal** values of the signals
3. Determine **uncertainties** for the test variables (signals)
4. If appropriate, write an **equation** that determines the test result from the test variables.
5. Determine, how uncertainties **propagate** from the test variables to the test result.
6. Apply the known uncertainties to the propagation equation to calculate the **propagation of the uncertainties**
7. **Add all uncertainties** using the RSS equation into a total uncertainty value
Example of Error Propagation

A ruler is used to measure the dimensions of a cylinder and its volume is calculated from these dimensions. What is the uncertainty in the volume?

Measured Diameter $d=2''$
Measured Height $h=10''$
Ruler Resolution: $1/16''$