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Pauli's exclusion principle in spinor coordinate space

May 11, 2009

Abstract The Pauli exclusion principle is interpreted using a geometrical theory of electrons. Spin and spatial motion are described together in an eight dimensional coordinate system. The field equation derives from the condition of zero scalar curvature by identifying a scalar wave with a power of the conformal factor. The Dirac wave function is a gradient of the scalar in spin space. Electromagnetic and gravitational interactions are mediated by conformal transformations. An electron may be followed through a sequence of creation and annihilation processes. Two electrons may be described as branches of a single particle. Each satisfies a Dirac equation, but together they are a solution of the curvature condition. As two so identified electrons approach each other, the exclusion principle develops from the boundary conditions in spin space. The gradient motion does not allow the particles to overlap. Since the the spin-gradient of the scalar wave function is odd in the spinor-space coordinates, the sign of the wave function must change at the electron-electron boundary. The exclusion principle becomes geometry intrinsic and all electrons are combined into one field. Further applications are proposed including the possibility of improved numerical calculations in atomic and molecular systems. There also may be extension to nuclear or particle physics. Implications are expected for the properties of rotating objects in a gravitational field.

Keywords geometry · Pauli exclusion principle · Dirac equation

PACS 11.15.q · 12.10-g · 14.60.cd

Mathematics Subject Classification (2000) 53Z05 · 81S99 · 83E15

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1 Introduction

The Pauli exclusion principle has been an enigmatic element of quantum mechanics from the start. Theoretical developments (1) support the experimentally observed effects, but the explanations have always been considered unintuitive and mathematically intricate. Quantum field theory (2; 3), argues that this effect is an essential property of fermions, but the development has always been in flat space and is not compatible with the curved space of general relativity. Even so, the interrelationship of spin and statistics is expected to hold. Elementary attempts to understand the effect have not been convincing, nor do they have a well defined relation to formal field theory.

Recently, new curved space theories have become sufficiently well developed to address the issues of spin one-half particles in a quantum setting (4). This approach has been developed from earlier constructions of a covariant scalar quantum field (5; 6). The theoretical approach is however entirely different from the usual formulation of quantum field theory. The structure of quantum mechanics itself is addressed, using differential geometry, in a way similar to general relativity. The resulting generalization of the Dirac equation allows the introduction of other properties that are known for electrons. It is an ongoing project and further developments are likely. In this process, the correspondence between the geometrical approach and standard field theory is not always apparent. Advantages include the integration of gravity and the avoidance of inconsistencies when interactions are included (7). Efforts to date have addressed particles individually. To understand exclusion, more than one particle must come into the description. It has been found possible to do this using the spinor space.

The discussion is intuitive and uses a minimum of mathematics. There is a short explanation of geometrical theories, followed by an exposition of spinor coordinates and then the exclusion principle.

2 Geometry and Quantum Mechanics

There is a basic mathematical problem that comes from the historical development of quantum theory. The commutation relations for a single particle, as expressed in Heisenberg's matrix mechanics, have a validity independent of any choice of final operand.

$$\mathbf{pq} - \mathbf{qp} = -i\hbar \quad (1)$$

The intention was to maintain the original classical equations, but to reinterpret the variables as operators (q-numbers) rather than as parameters (c-numbers). With the progression to wave mechanics (8; 9; 10), the matrix equations were rewritten using differential operators applied to a function. This mapping has been successful only for flat space-time, when the operator identity

$$\left[\frac{\partial}{\partial q} q - q \frac{\partial}{\partial q} \right] \psi = \psi. \quad (2)$$

associates the derivative with the Heisenberg momentum.

In a covariant setting, the use of operators as physical quantities has led to considerable mathematical difficulty. These operators are fragments of differential

equations. If the operator p_μ is to be a physical quantity in a curvilinear theory, it must transform covariantly. Furthermore, it must also produce covariant results when acting as an operator. Covariant derivatives

$$D_j \Phi^i = \Phi^i_{;j} = \frac{\partial \Phi^i}{\partial x^j} + \Gamma_{jk}^i \Phi^k \quad (3)$$

cannot be extended to bare operators. There is always trouble in higher order. It is usually claimed that if the energy is identified correctly, the quantization that results from operator substitution will give the correct result. However, when general relativity is combined with quantum mechanics, no satisfactory definition of this energy is available. The operator calculus, as might be expressed in terms of covariant derivatives, gives ambiguous results when applied to the different types of operands, vector, tensor or spinor. The quantization program encounters unmitigating difficulties and remains in jeopardy (11).

A wave packet in the curved geometry of four dimensional space-time can be split by the gravitational field. Mono-energetic states are preempted by dispersion and scattering. The quantum theory of a single particle will not have a classical limit. More specifically, the integrals used to define the Fourier transform cannot be extended to infinity. The non-locality of the energy-momentum operator conflicts with the limited extent of the local frame.

Different continuations may be available to resolve the contradictions. For the discussion at hand, bare operators are rejected as valid ontological quantities and physical propositions are formulated exclusively with completed differential equations. This assures that the ambiguities that may be associated with the covariant definition of a bare operator have all been resolved and will not interfere with theoretical constructions. In this formulation, a formal transition from classical physics to quantum physics is not used. The quantum behavior is an inherent part of an appropriately chosen geometry.

Following the arguments of Veblen (12), covariant equations are to come from the Riemann tensor. Thus, to satisfy equivalence, quantum waves must be constructed in this way. It is known that a linear wave equation can be derived from the behavior of the conformal factor. To see this, let the conformal factor in n dimensions be denoted by ω , and use the substitution $\omega = \Psi^p$ with $p = 4/(n-2)$. Setting the scalar curvature to zero produces the required properties and identifies Ψ as a possible physical quantity.

$$\frac{\partial^2 \Psi}{\partial x^a \partial x_a} \equiv R = 0 \quad (4)$$

The field Ψ may be used as a wave function.

The five dimensional case forms the basis of a theory of scalar quantum particles. In actuality, there are no stable particles of this type but it is a useful model showing how to construct a quantum-gravitational field. Classical five dimensional theory provides much of the mathematics and the quantum identifications are now known. Electromagnetic and gravitational fields are introduced through the metric tensor. The particles are geometrical quantities that obey a conformal wave equation. When expanded, electromagnetic and gravitational forces are implicitly part

of a linear quantum field equation.

$$\frac{1}{\sqrt{-\dot{g}}}\left(i\hbar\frac{\partial}{\partial x^\mu} - eA_\mu\right)\sqrt{-\dot{g}}g^{\mu\nu}\left(i\hbar\frac{\partial}{\partial x^\nu} - eA_\nu\right)\psi = \left[m^2 + \frac{3}{16}\left(\dot{R} - \frac{e^2}{4m^2}F_{\alpha\beta}F^{\alpha\beta}\right)\right]\psi. \quad (5)$$

The term in $F_{\alpha\beta}F^{\alpha\beta}$ is generally present in five-covariant theories. The production of particle pairs is affected. Sharp corners, as usually visualized, are softened because of the electromagnetic contribution to the effective mass. Experimental tests of the effect of this term remain open.

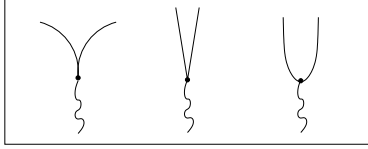


Fig. 1 Pair production is visualized as a continuous process in five-space. Particles connect with their anti-particles smoothly.

Source terms can also be developed. Particle interactions are mediated only by conformal transformations. It is an unproven conjecture that this is required by the mass-energy equivalence of any interaction. Forces are always defined through the conformal parameter as expressed relative to a flat space.

$$R_{ij}(\omega\gamma^{mn}) = 0 \quad (6)$$

implies

$$R_{ij}(\gamma^{mn}) = T_{ij}. \quad (7)$$

For the five dimensional case, the Einstein field equations are generated with corrections for quantum effects.

$$R^{\alpha\beta} = 8\pi\kappa \left[F^\alpha_\mu F^{\mu\beta} + m|\psi|^2 \frac{e^2}{m^2} A^\alpha A^\beta + m|\psi|^2 \frac{1 - (e^2/m^2)A^2}{2 - (e^2/m^2)A^2} g^{\alpha\beta} \right] \quad (8)$$

The new components may be related to what is now called dark energy. The Maxwell equations become

$$F^{\beta\mu}{}_{|\mu} = 4\pi e|\psi|^2 A^\beta \quad (9)$$

and have the correct quantum current.

3 Second quantization

The conventional formalism of second quantization defines the behavior of fermions and bosons in terms of operator commutator relations (13). The similar formalism used for the two cases is not extended into the geometrical representation. There is no direct correspondence. A Green's function, as used for a propagator, may be found from the differential equation, but is not an ontological starting point. For a given exact geometry, exclusion arises from the underlying differential equations.

For photons and gravitons, and presumably other fundamental bosons, a time symmetric formulation meets other needs of the geometry. At present, the development of the time-symmetric theory has arrived only to the level of quantum electrodynamics. In this formulation, the fundamental vector potential produced by a source particle, is always chosen as half the sum of the advanced and retarded solutions.

$$A_\mu = \frac{1}{2}(A_\mu(\text{ret.}) + A_\mu(\text{adv.})) \quad (10)$$

For second quantization, this construction avoids explicit free fields. The free interactions come from distant particles—the ‘absorber’. This point of view has been worked out in some detail (14; 15; 16; 17). It is a great simplification in principle because without explicit free fields, there is no field quantization. All interactions stem from source particles. For quantum radiation, it is equivalent to assigning the discrete levels of the black body to the states of the cavity walls. All electromagnetic fields are attached to quantized emitters, and no quantum structure is imposed on the interacting wave fields. The correct statistics are automatically

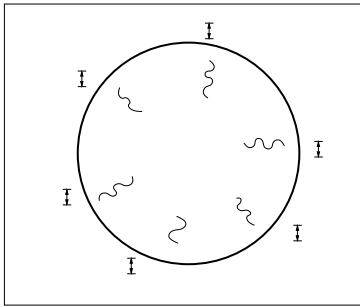


Fig. 2 The quantization of the electromagnetic field in a cavity is equivalent to the quantization of the oscillators in the wall.

produced and bare operators are avoided. A similar equivalent construction is presumed to be possible for gravity. All gravitons are attached to their respective source particles. It is worth mentioning that in the five dimensional theory, these electromagnetic and gravitational interactions are equivalent. The geometrical form of the interaction subsumes the role of creation and annihilation operators.

The statistical properties of electrons are the primary subject of this workshop. The effects were first observed in the specific heat of a monatomic gas (18), and the alternation of intensities of the rotational lines in diatomic molecules (19). Usually, it is the task of creation and annihilation operators to perform the book-keeping .

$$\{b_\alpha, b_{\alpha'}\} = 0 \quad (11)$$

$$\{b_\alpha^\dagger, b_{\alpha'}^\dagger\} = 0 \quad (12)$$

$$\{b_\alpha, b_{\alpha'}^\dagger\} = \delta_{\alpha\alpha'} \quad (13)$$

These are the last of the essential operators of physics that must be brought into geometry. Now, it is well known that the anti-commutation relations do not originate

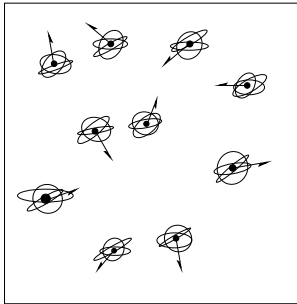


Fig. 3 Statistical properties of Fermi particles were first identified in the heat capacity of a monatomic gas or the alternation of line intensities in the spectra of diatomic molecules.

from any part of classical physics. They are not derivable from Poisson brackets during quantization. A different approach is needed and electron spin must be included. Fortunately, a suitable geometry for spin has always been available.

4 Spinor coordinates

Spinor coordinates (6) are characterized by the property that they obey, locally, the spinor transform law. It is a simple idea, implicit in Cartan's books (20), but possibly also considered by others, such as van der Waerden (21) or perhaps Pauli himself. The author is unaware of any other attempts to use such a coordinate system. Initially, one electron is assigned to spinor space with eight coordinates that combine into complex pairs.

$$\xi^A = \xi_r^A + i\xi_i^A \quad \text{and} \quad \xi^{\bar{A}} = \xi_r^A - i\xi_i^A \quad \text{for } A = 1 \cdots 4. \quad (14)$$

The spinor metric is diagonal

$$\varepsilon_{A\bar{B}} = \varepsilon^{A\bar{B}} = \text{diag}(1, 1, -1, -1). \quad (15)$$

and is easily written in terms of the complex infinitesimals

$$d\sigma^2 = \varepsilon_{A\bar{B}} d\xi^A d\xi^{\bar{B}}. \quad (16)$$

It is proper to use $\varepsilon_{A\bar{B}}$ to form the adjoint of a complex wave function rather than γ^0 (22).

The relation to space time is given as an equation in infinitesimals. The coefficients follow from the rules for five dimensional extended Lorentz transformations. The Dirac matrices combine displacements in spinor space to give elements in space-time. It must be of the form

$$dx^m = \zeta^A \gamma_A^m{}^B d\xi^{\bar{C}} \varepsilon_{\bar{C}B} + d\xi^A \bar{\gamma}_A^m{}^B \zeta^{\bar{C}} \varepsilon_{\bar{C}B} \equiv \zeta \gamma^m d\xi^\dagger + d\xi \gamma^{\dagger m} \zeta^\dagger. \quad (17)$$

where ζ^A is a spinor used to define the local orientation of the spinor space in extended space-time.

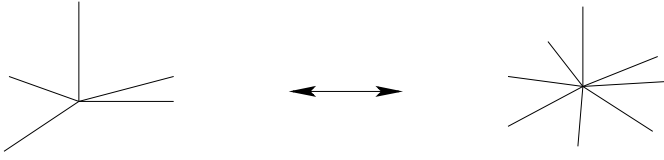


Fig. 4 The complex four dimensional spinor space is mapped to space-time by using the five anti-commuting Dirac spinors.

Following the development of conformal waves in five dimensions, an analogous construction is used in eight dimensions to produce the quantum wave equation.¹

$$0 = \boxtimes \Psi \equiv \varepsilon^{\bar{A}B} \frac{\partial}{\partial \xi^{\bar{A}}} \frac{\partial}{\partial \xi^B} \Psi \equiv \varepsilon^{\bar{A}B} \frac{\partial \Psi_B}{\partial \xi^{\bar{A}}}, \quad (18)$$

Not every solution of (18) is an electron. Other conditions are applied to insure that it has a well defined rest mass and a complete set of spin characteristics. The scalar wave-function, Ψ is presumed to be a function of extended space-time, $x^m \equiv (x^\mu, \tau)$ and has dependence on the fifth coordinate, $\exp(-im\tau)$. The Dirac spinor is formally defined as the spinor-space gradient of the scalar wave

$$\Psi_B = \frac{\partial \Psi}{\partial \xi^B}. \quad (19)$$

With these choices, the chain rule and the transformation (17) result in

$$\zeta^D \left[\gamma_D^m{}^B \frac{\partial \Psi_B}{\partial x^m} \right] = 0. \quad (20)$$

Allowing the multiplying factor ζ^D to be arbitrary requires the term in brackets to be zero. This reduction of the conformal equation is exactly the Dirac equation as it is usually written in five dimensional form. Different spin orientations are implied by the choice of Ψ and ζ .

In the idealized case of no interactions, a particle is a plane wave having a particular direction in spinor space.

$$\Psi = e^{i(\mathbf{k}\mathbf{x} - \omega t - m\tau)} \equiv e^{ik_l x^l}, \text{ summed for } l = 0 \dots 4 \quad (21)$$

with

$$k_l = (\mathbf{k}, \omega, m) \quad (22)$$

The Dirac wave function may be found by spinor space differentiation.

$$\Psi_A = \Psi ik_l \frac{\partial x^l}{\partial \xi^A} = i\Psi k_l \gamma^{+l} \zeta^\dagger \quad (23)$$

$$= i\Psi \begin{pmatrix} k_0 & 0 & im - k_3 & -k_1 + ik_2 \\ 0 & k_0 & -k_1 - ik_2 & im + k_3 \\ im + k_3 & k_1 - ik_2 & -k_0 & 0 \\ k_1 + ik_2 & im - k_3 & 0 & -k_0 \end{pmatrix} \zeta^\dagger \quad (24)$$

¹ The octagon, as a symbol of the eight dimensional D'Alembertian, is reflected onto itself to distinguish it from a circle.

The choice of polarization comes from the frame orientation spinor. For more complicated spinor wave functions this plane wave solution may be used as a linear approximation in a local region. Rotations of the wave function in spinor space follow crossing symmetry as electrons are converted to positrons or neutrinos.

5 Transformation theory of interaction

It is assumed, as in five space, that all particle interactions are mediated by conformal transformations. The electromagnetic and gravitational effects are identified by the anti-commutation law of Dirac spinors. The gravitational and electromagnetic interactions are inferred in the spinor space from the equivalence of conformal transformations in five and eight dimensions. Both the five and eight dimensional spaces are assumed to be conformally flat.

$$\frac{1}{2}\{\gamma^m, \gamma^n\} \equiv \frac{1}{2}(\gamma^m \gamma^n + \gamma^n \gamma^m) = \gamma^{mn} \quad (25)$$

with

$$\gamma_{mn} \equiv \begin{pmatrix} g_{\mu\nu} - \mathcal{A}_\mu \mathcal{A}_\nu & -\mathcal{A}_\mu \\ -\mathcal{A}_\nu & -1 \end{pmatrix} \quad (26)$$

Both rotations and dilatations may be used. Beginning in flat space with a plane wave particle, successive transformations can be performed to give a particular gravito-electromagnetic interaction. Weak interactions may occur in the spinor space but cannot be coupled to transformations in extended space-time.

6 An identified pair of electrons

Now, we can begin to consider the exclusion forces. In the series of vignettes of figure (5), successive transformations of an electron system are displayed as they might occur for increasingly stronger electromagnetic fields. Starting with a low velocity ‘packet’ state, a progression of conformal transformations increases the velocities producing a motion that includes the production and annihilation of pairs in high fields. In subsequent frames, the time reversals are moved earlier and later, eventually leaving two electrons and one positron. The positron can also be moved away from the experimental area, leaving only the two electrons. These are actually the same particle. We will call them an “identified pair”. Both electrons satisfy the Dirac equation at each step. The system satisfies equation (18). The diffeomorphic transformations connect continuously through to the final frame.

6.1 Parallel spins

As the full effect of external gravitational and electromagnetic fields is available, an identified pair may be prepared with different space and spin states. The case of parallel spins is easiest. Because of the Pauli principle, we would expect the two electrons to be disjoint in space time. Since a single point in spinor space maps to a single point in space-time, the electrons must also be separated in spinor space.

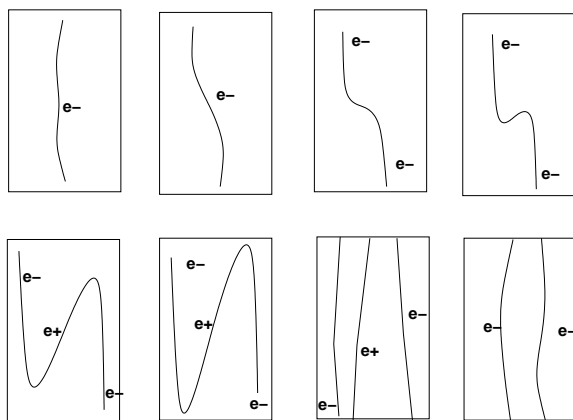


Fig. 5 An electron pair can be produced as a combination of electromagnetic fields mediated by conformal transformations. These electrons may actually be connected through a positron that participates in processes of pair creation and annihilation.

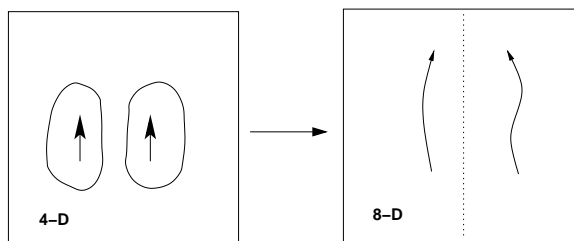


Fig. 6 Two electrons, with parallel spins are disjoint in space-time.

6.2 Anti-parallel spins

It is now to be argued that electrons with non-parallel spins are also disjoint in spinor space. In this case, such as might occur in the ground state of helium, several steps are needed. By experimental procedures, the electrons may be separated—the helium will be ionized. The identified pair may then be brought into adjacent regions with parallel spins. At this point, the electrons cannot be overlapping. Integrating backwards, it is easy to see that the electrons remain separated and cannot be made to overlap at any earlier time. The flow lines cannot cross. In any case, once it is known that the electrons are separate at some particular time, no solution will allow them to overlap and develop a some common volume. They must have always been separated. In practice, any electron can be removed from the assembled system by electromagnetic forces. Apparently, then, all electrons remain disjoint in spinor space.

7 Spinor space wave propagation

Now it is to be shown that the mutual exclusion of electrons in spinor space is determined from the geometry. As an identified pair, the two electrons propagate according to the equation (18), which is also the Dirac equation for each one separately. As they approach and meet, the Pauli principle must come into play.

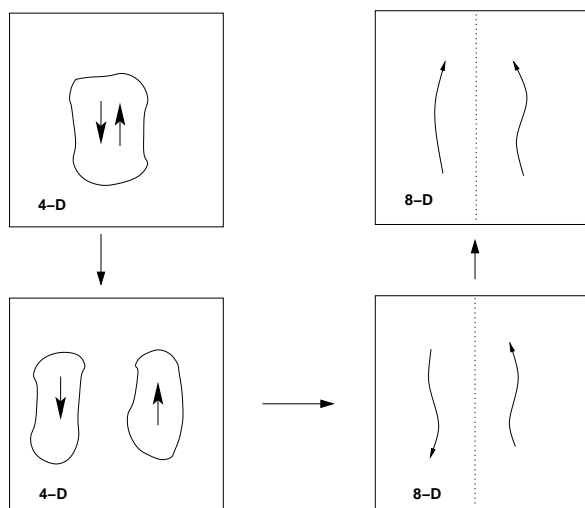


Fig. 7 Two electrons with non-parallel spins may occupy the same point in space-time. However, a sequence of electromagnetic interactions will, in general, separate the two electrons. For the separated electrons, the parallel space argument holds. The electrons must have separate wave functions in spinor space.

The Dirac wave equation dictates the time evolution until the moment of contact. All exclusion effects occur at this boundary. The process can be visualized by using the Huygens-Fresnel-Feynman type construction. Electrons, of course, may continue to overlap in space-time, but will have, at most, a common boundary in spinor space.

At the instant of contact, the boundary condition begins to affect the electrons. Working with the wave front formalism, contributions of one particle to its own new wavefront have a plus sign while contributions from across the developing boundary have a minus sign. As the particle separation goes to zero, opposite signs are forced.

$$\psi'(1) = a[\psi(1) - \psi(2)] \quad (27)$$

$$\psi'(2) = a[\psi(2) - \psi(1)] \quad (28)$$

The developed constraint propagates away from the boundary through each particle, according to the respective Dirac equation. As time goes on, the zero of the boundary is maintained and moves to keep the slopes equal and opposite. $\psi'(1)$ must equal minus $\psi'(2)$ for small equal displacements away from the zero crossing. The exchange anti-symmetry is fully developed.

Now, from the point of view of the wave equation in spinor coordinates, this is a completely natural occurrence. The two electrons actually obey the equation (18) and must have a smooth first derivative at the boundary. At the spinor boundary, the wave function for both particles is just equation (19) and must be exactly anti-symmetric for the infinitesimal coordinate-spin interchange. The common wave function is always odd at a meeting point.

This analysis can be extended to more particles. A collection of electrons may begin in spinor space as disjoint objects without any common boundary. They will move and expand. The anti-symmetrical boundary condition is developed at any junction. In regions away from the boundaries, the Dirac equation prevails.

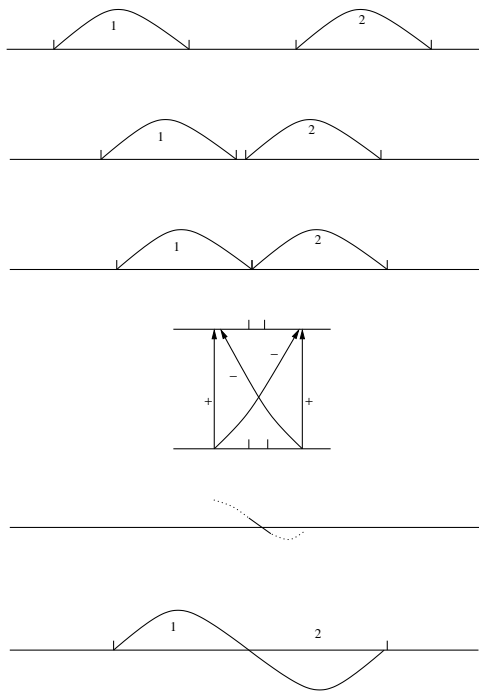


Fig. 8 Two electrons approach each other in spinor space. The motion is entirely determined by their respective Dirac equations. As they meet in spinor space, the exchange condition insures that an anti-symmetric boundary conditions develops. Only boundary points are in common. The boundary constraint propagates through each electron according to the Dirac equation.

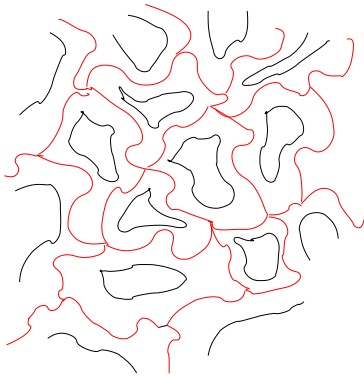


Fig. 9 Multiple electrons, each a solution of the Dirac equation may meet in spinor space. the anti-symmetric boundary condition develops at the common boundaries.

Anti-symmetry remains in force. The Pauli exclusion principle is the result of the properties of this global differential equation. Pauli exclusion is equivalent to the assumption that all electrons are a collective solution of the conformal wave equation.

8 Ongoing considerations

These ideas are new. Sometimes the applications can make a contribution to understanding. A few possibilities come to mind.

Computational advantages: First, there may be some simplification when calculating multiple electron systems. For the usual construction the computer array size must increase as $N!$ to allow for the Fock space anti-symmetrization (23; 24). Apparently, such large dimensional requirements may now be simplified since it is predicted that all of the electrons can be included in a single eight dimensional field. With each additional electron, more points will be needed but with the total increasing only as N . Atomic or molecular systems that normally would exceed computational capabilities may become accessible to exact calculations. For the stationary state of a multi-electron system, without time dependence or weak interactions, the requirements may even be simpler.

Relativistic formalism: As the formalism developed here is not based on an expansion in orders of interaction, it should be relativistic and exact, including any energy or combination of interactions. Quantum electrodynamics should appear as a limiting case. What are the exact relationships to other electron theories including the transmutation to neutrinos? What is the relationship to the introduction of anti-symmetrical exchanges in the evaluation of Feynman diagrams?

Self interactions: The characteristic relationship of particle-particle interactions to self-interactions needs to be clarified as it may be related to calculations of the fine structure constant (25). In the usual formulation, an electron does not experience the forces of its own electromagnetic field even though two electrons of an identified pair must interact. What are the implied relationships between diffraction and electromagnetic repulsion, if any?

Pair production: The creation and annihilation operators of field theory are quite useful in calculations. What is the fundamental character of these operations? How are they related to relativity theory? Can electron creation operators be represented as diffeomorphic transformations of multi-electron wave functions in spinor space?

Other particles: How will other fermions appear? Do quarks have wave functions in spinor-space? Do all the fermions have wave functions in the same space, or is it necessary to use, a Fock space construction? What are the experimental tests that might distinguish these possibilities? Are there also benefits for the calculation of nuclear properties?

Rotating test particles: A number of problems involving rotation, persist in general relativity and quantum theory. How is the Dirac-Thirring paradox (6) to be resolved? What happens to Newton's rotating bucket of water if the bucket dimensions are stabilized with the Pauli principle? What can be said about the theories of Mathisson, Corinaldesi, and Papapetrou (26; 27; 28; 29) that involve the motion of spinning objects in gravitational fields? Is there a fundamental mechanical difference between orbital and spin angular momentum? What are the experimental consequences? How is this related to general questions of spin rotation in quantum theory (30).

9 Conclusion

This analysis depends on the use of a geometrical description of fundamental physics. Techniques of differential geometry are essential for the required manipulations of the electron wave functions. In the context of the combined effects of diffraction, gravitation, electromagnetism, and weak interactions, the spinor coordinate system has natural relevance. This elementary description of the Pauli exclusion principle is presented as an addition to the other known capabilities of this formalism.

For geometrical physics, this construction is of crucial importance. To arrive at a complete theory, the algebra of second quantization must be assigned to a differential system. Here, the multiplicity of metrics used for particles in the five dimensional formalism is replaced by a single piece of eight dimensional fabric that includes exclusion. With these developments, a number of ongoing issues can be addressed and may bring new answers.

Acknowledgements This article is based on a talk given at “Spinstat2008” in Trieste, Italy. I would like to thank the organizers for a well planned and effective meeting.

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