Abstract—Conventional amplitude-phase modulations (APMs) are designed such that the minimum Euclidean distance between constellation points is maximized. Unlike these modulations, the error performance of spatial modulation (SM) is not only a function of the Euclidean distances but also the energy of each symbol, i.e. constellation point. Therefore, conventional APM schemes that are designed solely based on the notion of Euclidean distance are not necessarily suitable for SM transmission. In this paper, the constellation design for SM is investigated and it is shown that, by a proper constellation design, a significant performance gain is obtained compared to the well-known phase shift keying (PSK) or quadrature amplitude modulation (QAM).

Index Terms—multi-input multi-output; spatial modulation; amplitude-phase modulation.

I. INTRODUCTION

The advent of multi-input multi-output (MIMO) communication systems has enabled new modulation domains (e.g. space and code) in addition to widely used conventional domains such as amplitude-phase (i.e. inphase-quadrature) or frequency domains [1], [2]. Furthermore, these new domains have been used in conjunction with conventional domains to design more advanced modulation schemes. In this context, joint constellation design and optimal constellation breakdown between different domains have emerged as interesting yet demanding problems that call for efficient solutions [3].

Spatial modulation (SM) is one of these new modulation schemes that utilizes the amplitude-phase and space domains for data transmission. In SM, the index of the active transmitting antenna along with the inphase-quadrature components of transmit signal are used to map the information [4]–[6]. One interesting and important question is the problem of optimal signal design for SM. In [3], optimal constellation breakdown for SM is studied, and it is shown that at any fixed transmission rate, there exists an optimal amplitude-phase modulation (APM) size as well as the number of transmit antennas that minimize the symbol error rate (SER). The optimal SM breakdown for the two common types of modulations, i.e. phase shift keying (PSK) and quadrature amplitude modulation (QAM), has been derived. Nevertheless, as far as the optimal design of SM is concerned, applying PSK and QAM schemes to SM does not necessarily result in the minimum SER. In fact, it is shown that the union bound of the SER of SM is the function of the Euclidean distance between the APM constellation points as well as the norm of the symbols [3], [5], [6]. This is in sharp contrast to the notion that the performance of a modulation scheme is mainly a function of the minimum Euclidean distance.

Star-QAM is a quadrature amplitude modulation scheme which allows differential encoding/decoding and increases bandwidth efficiency in Rayleigh fading channels [7], [8]. In [9], it is shown that space-time shift keying (STSK) using star-QAM outperforms STSK with conventional APM schemes. In [10], star-QAM is considered as a suitable candidate for SM, and a low-complexity algorithm is introduced to design twin-ring star-QAM to minimize the SER. In this paper, we consider the design of APM schemes that are suitable for SM in SER sense. We consider the general problem of APM design for SM without imposing a predefined structure on the constellation diagram. We set up an optimization problem, and, by solving it numerically, show that a generalized star-QAM scheme is indeed a good candidate for SM. While restricting structures, such as two-ring 16 star-QAM with equal number of constellation points in each ring, are considered for star-QAM in the literature, we consider the more general problem without imposing any limitation on the number of rings or the composition of the constellation points in different rings. Although the APM design, in the most general case, is obtained numerically, the impact of different system parameters such as the number of transmit and receive antennas on the structure of optimal APM scheme for SM are mathematically analyzed. More precisely, by analysis we show that in special cases of large number of transmit or receive antennas, the optimal constellation converges to the well known PSK and square-QAM (SQAM), respectively.

The remainder of the paper is organized as follows. In Section II, we introduce the system model for SM transmission. In Section III, we present the optimization problem for APM design and by mathematical analysis, we evaluate the effect of different system parameters on the proposed constellation design. In Section IV, we present some numerical results, and finally, Section V concludes the letter.

II. SPATIAL MODULATION

The received signal for an SM system with $N$ transmit and $M$ receive antennas can be written as

$$y = \sqrt{\gamma} H_{s} x_{n} + v, \quad n \in \{1, \ldots, N\}, \quad l \in \{1, \ldots, L\} \quad (1)$$
where $\gamma$ is the signal-to-noise ratio (SNR), $s_l$ is an $L$-ary APM symbol, $v$ is the zero-mean complex additive white Gaussian noise vector with unitary covariance matrix, $x_n$ is the $n$-th SSK constellation vector with one nonzero element equal to “1”, and $H$ is the $M \times N$ uncorrelated Gaussian MIMO channel matrix with unit variance entries.

The union bound of the average SER of SM at high SNRs can generally be expressed as [3]

$$P_M (L, N) \approx \left( \frac{2M - 1}{M} \right) \gamma^M \left( \frac{N - 1}{N} \right) \hat{P}_M (L) + \bar{P}_M (L)$$

(2)

where

$$\hat{P}_M (L) = \sum_{l=1}^{L} \left( |s_l|^2 + |s_{l'}|^2 \right)^{-M}$$

(3)

and

$$\bar{P}_M (L) = \sum_{l=1, l' = 1 \atop l \neq l'}^{L} |s_l - s_{l'}|^{-2M}.$$ 

(4)

Note that (4) is related to the probability of error corresponding to the APM. In conventional modulations (i.e. when $N = 1$), (3) is absent in the union bound equation, and therefore, only Euclidean distances between APM symbols affect the error performance of such modulation schemes. The impact of the $\hat{P}_M (L)$ term on the performance of SM increases by increasing the number of transmit antennas [3]. As a result, in addition to the Euclidean distances between APM symbols, and especially at large number of transmit antennas, the amplitudes of the APM symbols play an important role in the performance.

III. APM Constellation Design for SM

In this section, we introduce the optimization problem to design APMs for SM based on minimizing the union bound of the probability of error in (2). Assuming $s_l = r_l \exp(j \theta_l)$, where $r_l$ is the amplitude and $\theta_l$ is the phase of the symbol, (3) and (4) can, respectively, be rewritten as

$$\hat{P}_M (L) = \sum_{l=1}^{L} \left( r_l^2 + r_{l'}^2 \right)^{-M}$$

(5)

and

$$\bar{P}_M (L) = \sum_{l=1, l' = 1 \atop l \neq l'}^{L} \left( r_l^2 + r_{l'}^2 \right)^{-M} (1 - \varepsilon_{l,l'} \cos(\theta_l - \theta_{l'}))^M$$

(6)

where $\varepsilon_{l,l'} = \frac{2r_l r_{l'}}{r_l^2 + r_{l'}^2}$. Note that $\bar{P}_M (L)$ does not depend on the phases of the symbols, while $\hat{P}_M (L)$ depends on the phases. For a fixed $L$, the optimization problem can be expressed as

$$\hat{\mathbf{d}} = \arg\min_{\mathbf{d} = \{r_1, ..., r_{L/4}, \theta_1, ..., \theta_{L/4}\}^T} \left\{ (N - 1)\hat{P}_M (L) + \bar{P}_M (L) \right\}$$

s.t. $\frac{1}{L} \sum_{l=1}^{L} r_l^2 = 1.$

(7)

The optimization problem in (7) indicates that the optimal APM scheme can vary for different number of transmit antennas. Intuitively, increasing the distances between the APM symbols as much as possible minimizes the second term in the right side of (7), while equalizing the amplitude of different symbols will result in the minimization of the first term. Finding a solution to (7) can be interpreted as making a balance between these two terms for different number of transmit antennas $N$.

To set up a tractable optimization problem, we make some valid assumptions. Based on the symmetry of $\hat{P}_M (L)$ and $\bar{P}_M (L)$ with respect to real and imaginary parts of symbols, we conjecture that the optimal constellation diagram is symmetric along both inphase and quadrature axes. By this assumption, we can modify the optimization problem as follows

$$\hat{\mathbf{d}} = \arg\min_{d = \{r_1, ..., r_{L/4}, \theta_1, ..., \theta_{L/4}\}^T} \left\{ (N - 1)\hat{P}_M (L) + \bar{P}_M (L) \right\}$$

s.t. $\frac{4}{L} \sum_{l=1}^{L/4} r_l^2 = 1.$

(8)

Also, $\hat{P}_M (L)$ and $\bar{P}_M (L)$ can be rewritten as

$$\hat{P}_M (L) = 16 \sum_{l=1, l' = 1}^{L/4} \left( r_l^2 + r_{l'}^2 \right)^{-M}$$

(9)

and

$$\bar{P}_M (L) = 2^{2-M} \sum_{l=1, l' = 1}^{L/4} \left( r_l^2 + r_{l'}^2 \right)^{-M} \delta_{l,l'} (M)$$

(10)

where

$$R_M (x) = \left( \frac{1 - x}{2} \right)^{-M} + \left( \frac{1 + x}{2} \right)^{-M}, \quad |x| < 1$$

(11)

and

$$\delta_{l,l'} (M) = \begin{cases} R_M \left( \chi_{l,l'}^+ \right) + R_M \left( \chi_{l,l'}^- \right), & l \neq l' \\ R_M \left( \chi_{l,l'}^+ \right) + 1, & l = l' \end{cases}$$

(12)

where $\chi_{l,l'}^\pm = \varepsilon_{l,l'} \cos (\theta_l \pm \theta_{l'})$. Reformulating (7) as (8) expedites the execution of the numerical optimization algorithm by reducing the number of variables by a factor of 4. The rest of the symbols can be simply obtained by

$$s_{1+L/4} = s_1^*; \quad s_{1+L/2} = -s_1; \quad s_{1+3L/4} = -s_1^*; \quad l = 1, ..., L/4$$

(13)

where $(\cdot)^*$ stands for the conjugate operation. While obtaining closed-form solutions for the optimization problem in (8) is very difficult, we solve it numerically for several cases in Section IV to find out suitable APMs and we show that depending on different parameters, such as the number of transmit and receive antennas, the optimal APM constellation approximately resembles a star-QAM constellation with different number of rings. In the sequel, we investigate the effect of different parameters on the structure of the optimal constellation diagram.
optimization problem in (7) can be approximated as
transmit antennas, PSK is the optimal APM scheme for SM.
for SM. This can be concluded from the
second term can in (7), however, be considered
can be neglected to find the optimal amplitudes of different
symbols. The second term can in (7), however, be considered
to find the optimal phases. This can be concluded from the
following Proposition.

**Proposition 1:** Asymptotically for very large number of
transmit antennas, PSK is the optimal APM scheme for SM.

**Proof:** For very large number of transmit antennas, the
optimization problem in (7) can be approximated as

\[
[
\hat{\theta}_1, ..., \hat{\theta}_L
]^{T} = \arg \min \hat{P}_M (L) \quad s.t. \quad \frac{1}{L} \sum_{l=1}^{L} r_l^2 = 1
\]

\[0 < r_l < \sqrt{L}\]

and

\[
\left[\theta_1, ..., \theta_L\right]^{T} = \arg \min \left[\theta_1, ..., \theta_L\right]^{T} : \quad 0 \leq \theta_l < 2\pi
\]

\[
\left[\begin{array}{c}
0
\end{array}\right]^{T}
\]

where \(\lambda\) is the Lagrange multiplier. By taking the derivative of \(\Delta\) with respect to \(r_l\), \(l = 1, ..., L\), and setting it to zero, we have

\[
\sum_{l'=1}^{L} (\hat{r}_l^2 + \hat{r}_{l'}^2)^{-M-1} = \frac{\lambda}{2M}, \quad l = 1, ..., L.
\]

Therefore, for any \(l\) and \(l'\) (\(l \neq l'\)), we have

\[
\sum_{l'=1}^{L} (\hat{r}_l^2 + \hat{r}_{l'}^2)^{-M-1} = \sum_{l'=1}^{L} (\hat{r}_l^2 + \hat{r}_{l'}^2)^{-M-1}
\]

\[
\left(\hat{r}_l^2 - \hat{r}_{l'}^2\right) \sum_{l'=1}^{L} \left\{ Q_M \left(\hat{r}_l^2 + \hat{r}_{l'}^2, \hat{r}_l^2 + \hat{r}_{l'}^2\right) \right\} = 0
\]

\[
\Delta = \sum_{l=1}^{L} (r_l^2 + r_{l'}^2)^{-M} + \lambda \left(\sum_{l=1}^{L} r_l^2 - L\right)
\]

A. Effect of the number of transmit antennas

As the number of transmit antennas \((N)\) increases, the
impact of the portion of the error related to spatial domain
increases and the optimal APM scheme for SM diverges from
conventional APMs. For very large values of \(N\), the first
term in the right side of (7) is dominant and the second term
can be neglected to find the optimal amplitudes of different
symbols. The second term can in (7), however, be considered
to find the optimal phases. This can be concluded from the
following Proposition.

**Proposition 1:** Asymptotically for very large number of
transmit and receive antennas, PSK is the optimal APM scheme for SM.

\[
N = 1
\]

\[
N = 16
\]

\[
N = 64
\]

\[
N = 1024
\]

Fig. 1. The proposed 64-ary constellation diagram of SM for different number of transmit and receive antennas.

\[
\hat{\theta}_1, ..., \hat{\theta}_L
\]
where \( Q_M(x, y) = \sum_{i=0}^{M} x^i y^{M-i} \). Since for any \( x, y > 0 \), \( Q_M(x, y) \) is positive, \( r^*_l = 1, l = 1, \ldots, L \), satisfies (19). Based on this fact and to minimize \( \bar{P}_M(L) \), PSK is the optimal APM scheme for SM with very large number of transmit antennas.

**B. Effect of the number of receive antennas**

Studying the impact of the number of receive antennas \((M)\) is not as straight forward as that of the transmit antennas. This is because \( M \) appears in as the exponent factor in \( P_M(L) \) and \( \bar{P}_M(L) \) in (5) and (6), respectively. To analyze the effect of \( M \), we express the following Proposition.

**Proposition 2:** By increasing the number of receive antennas, the ratio between \( \bar{P}_M(L) \) and \( \bar{P}_M(L) \) increases.

**Proof:** A direct and rigorous proof of this Proposition tends to be very difficult. However, several numerical trials verify the statement. Here, we introduce lower and upper bounds for the ratio of \( \bar{P}_M(L) \) and \( \bar{P}_M(L) \), and prove that both are increasing functions of \( M \). Considering (9) and (10), for this ratio we can write

\[
\frac{\bar{P}_M(L)}{\bar{P}_M(L)} = \frac{\sum_{l=1}^{L/4} (r^2 + r^2) - M \delta_{l,l'}(M)}{\sum_{l=1}^{L/4} (r^2 + r^2) - M}
\]

It is easy to show that for any positive series of \( \{a_i\} \) and \( \{b_i\} \) \((a_i, b_i \geq 0)\), we have

\[
\sum_{i=1}^{L/4} a_i \left( \sum_{i=1}^{L/4} b_i \right)^{-1} \leq \sum_{i=1}^{L/4} a_i \left( \sum_{i=1}^{L/4} b_i \right)^{-1} \leq \sum_{i=1}^{L/4} a_i \sum_{i=1}^{L/4} b_i
\]

Considering (20) and (21), one can conclude that

\[
\left( \sum_{l=1}^{L/4} \frac{2^{M+2}}{\delta_{l,l'}(M)} \right)^{-1} \leq \frac{\bar{P}_M(L)}{\bar{P}_M(L)} \leq \sum_{l=1}^{L/4} \frac{\delta_{l,l'}(M)}{2^{M+2}}
\]

As a property of \( R_M(x) \), we have

\[
R_{M+1}(x) - 2R_M(x) = x \left\{ \left( \frac{1-x}{2} \right)^{-M} - \left( \frac{1+x}{2} \right)^{-M} \right\} > 0
\]

Therefore, \( \delta_{l,l'}(M+1) > 2\delta_{l,l'}(M) \), and as the result both upper bound and lower bound in (22) are increasing functions of \( M \).

Based on Proposition 2, we can conclude that increasing the number of receive antennas \((M)\) increases the dominance of \( \bar{P}_M(L) \) over \( \bar{P}_M(L) \). This means that for large number of receive antennas, the optimal APM for SM is very close to conventional APMs such as QAM. Therefore in this case, the design of new APM schemes is unnecessary.

Note that as a conventional APM scheme such SQAM has a more favourable minimum distance property compared to PSK, according to Proposition 2, at a small \( N \), SQAM is generally a better choice for SM than PSK. While at a very large \( N \), based on Proposition 1, PSK is a better choice. Therefore, we expect that by increasing the number of transmit antennas, the optimal APM constellation diagram gradually changes from a constellation close to SQAM to PSK.

**IV. Simulation Results**

In this section, we provide some numerical results to investigate the APM design for SM. We also simulate the error performance of the obtained APM schemes and compare them with conventional modulation schemes.

We first use MATLAB Optimization Toolbox to find the solution to the optimization problem in (8). In Fig. 1, the constellation diagrams of proposed APMs with size \( L = 64 \) are illustrated for different numbers of transmit and receive antennas. As shown in Fig. 1, for \( N = 1 \), the numerically derived APM is very close to SQAM. However, as the number of transmit antennas increases, the proposed APM gradually diverges away from SQAM and toward PSK which is in
agreement with Proposition 1. Also, Fig. 1 demonstrates the effect of the number of receive antennas on the APM constellation diagram which verifies the validity of Proposition 2. As seen from the figure, the proposed numerically derived APM does not generally resemble well-known modulation schemes. However, upon further reexamining the figure, it turns out that the proposed APM constellation for different cases is close to star-QAM with varying number of rings and number of constellation points in each ring. Therefore, one can conclude that star-QAM can be considered as a suitable APM modulation scheme for SM. Adopting a structured modulation such as star-QAM has an added advantage of reduced detection complexity by using a constellation quantization (slicing) function [11] to avoid exhaustive search over different constellation symbols.

Note that unlike the star-QAM proposed in [10], the number of the constellation points in different rings of the proposed numerically derived star-QAM is not constant. In fact, the outer rings include more constellation points. Also in [10], a predetermined number of rings is considered for all the cases. However, as it is shown in Fig. 1, as the number of transmit antennas increases, the number of rings in the proposed constellation decreases, and asymptotically \((N >> 1)\) converges to one, i.e. PSK constellation.

We next compare the SER performance of SM with proposed APM with that of SM with PSK and QAM in Fig. 2 for \(L = 16, N = 256\) and \(M = 3\). As a benchmark, the performance of a single-antenna transmitter exploiting 64-QAM and 64-PSK is also shown. As shown in the figure, although conventional QAM outperforms PSK in single-antenna systems, PSK is a better choice for SM. In fact, as seen, the performance of SM with proposed APM is every close to that of SM with PSK in this case. This means that in this case, where the number of transmit antennas is large enough \((N = 256)\), SM with PSK matches the performance of optimal SM (see Proposition 1).

Fig. 3 shows the SER performance of SM for \(L = 64, N = 64\) and \(M = 2\). As illustrated in this figure, SM with proposed APM outperforms SM with QAM and PSK with 2dB and 5dB, respectively. On the other hand, unlike the previous case, QAM turns out to be a better constellation for SM compared to PSK. As a benchmark, the performance of single-antenna 16-QAM and 16-PSK modulations is also shown. Interestingly, the performance of single-antenna PSK is almost the same as that of SM with PSK. This is because, in this case, the error related to APM is dominant to the error related to space.

V. Conclusion

We investigated the design of APM schemes for SM. An optimization problem was first formulated, and the effect of the number of transmit and receive antennas on the structure of the optimal constellation was studied analytically. We showed that in two extreme cases of large number of transmit and receive antennas, the optimal constellation converges to SQAM and PSK, respectively. The optimization problem was then solved numerically, and the APM constellations were derived for different cases. It was argued that, in many cases, the numerically derived constellation closely resembles a star-QAM with different number of rings. With simulations, we showed that the SER performance of SM with the proposed APM is superior to that of SM with QAM or PSK.

References